The quartic equation: invariants and Euler's solution revealed [1]

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http://www.nickalls.org/dick/papers/maths/quartic2009.pdf

1 Introduction

The central role of the resolvent cubic in the solution of the quartic was first appreciated by Leonard Euler (1707–1783). Euler's quartic solution first appeared as a brief section (§ 5) in a paper on roots of equations [1, 2], and was later expanded into a chapter entitled *Of a new method of resolving equations of the fourth degree* (§§ 773–783) in his *Elements of algebra* [3, 4].

Euler's quartic solution was an important advance, in which he showed that each of the roots of a reduced quartic can be represented as the sum of three square roots, say $\pm\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3}$, where the r_i $(i = 1, 2, 3)$ are the roots of a resolvent cubic. A quartic equation in x is said to be *reduced* if the coefficient of x^3 is zero. This can always be achieved by a simple change of variable.

Motivated by the recent tercentenary of Euler's birth, this article describes the geometric basis underlying both the r_i and the sign of the product $\sqrt{r_1 r_2 r_3}$, these being two key aspects of Euler's solution. Finally, we reveal the beautiful dynamic between Euler's resolvent cubic and the quartic invariants G, H, I, J [5, 6, 7], and propose a new class of algebraic object.

2 Geometric basis for the r_i

A significant property of the reduced quartic equation is that the four roots can be completely defined using only three parameters. For example, let z_j $(j = 1, 2, 3, 4)$ be the roots (see Figure 1) of a reduced quartic equation,

$$Z(x) \equiv ax^4 + px^2 + qx + r = 0. \tag{1}$$

As the sum of the roots is zero (the coefficient of the cubic term is zero), it follows that we can define the points midway between z_1, z_2 and z_3, z_4 as $\pm g$. Let $z_2 - z_1 = 2\alpha$ and $z_4 - z_3 = 2\beta$. The four roots can then be expressed as follows:

$$\begin{cases} z_1, z_2 = -g \pm \alpha, \\ z_3, z_4 = +g \pm \beta. \end{cases}$$

Since specifying one pair of quartic roots necessarily defines the remaining pair, there are just three different ways of allocating the pairs of roots, each associated with its own g, α, β, the inter-relationship between which lies at the heart of a remarkable symmetry which underpins the solution of the quartic.

[1] This minor revision of the original article corrects typographic errors and incorporates some explanatory footnotes. The original published version is available from the JSTOR archive at http://www.jstor.org/stable/40378672.

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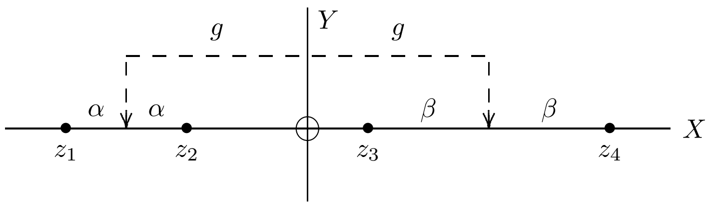

Figure 1:

For example if, with no loss of generality, we let

$$\begin{cases} z_3+z_4 &=& 2g_1, \\ z_3+z_1 &=& 2g_2, \\ z_3+z_2 &=& 2g_3, \end{cases} \tag{2}$$

then

$$\begin{aligned} 2(g_2+g_3) &= 2z_3+z_1+z_2, \\ &= (z_1+z_2+z_3+z_4)+z_3-z_4, \\ &= z_3-z_4=-2\beta_1, \end{aligned}$$

and similarly

$$2(g_2-g_3)=z_1-z_2=-2\alpha_1,$$

and hence

$$\begin{cases} \alpha_1=-(g_2-g_3), \\ \beta_1=-(g_2+g_3). \end{cases}$$

Thus the α_k, β_k $(k=1,2,3)$ are actually simple functions of the g_i $(i\neq k)$ such that each of the four roots z_j can be expressed as a function of the g_i alone, as follows[3]:

$$\begin{cases} z_1=-g_1-\alpha_1=-g_1+(g_2-g_3)=-g_1+g_2-g_3, \\ z_2=-g_1+\alpha_1=-g_1-(g_2-g_3)=-g_1-g_2+g_3, \\ z_3=+g_1-\beta_1=+g_1+(g_2+g_3)=+g_1+g_2+g_3, \\ z_4=+g_1+\beta_1=+g_1-(g_2+g_3)=+g_1-g_2-g_3. \end{cases} \tag{3}$$

Thus Euler's r_i are the same as the g_i^2.

3 Euler's resolvent cubic

Using these observations we can reconstruct a given reduced quartic equation, say Equation 1, which then leads to a resolvent cubic and hence to the solution. Let the roots of $Z(x)=0$ be $-g\pm\alpha$ and $g\pm\beta$ (Figure 1).

$$Z(x)\equiv\{x-(-g-\alpha)\}\{x-(-g+\alpha)\}\{x-(g-\beta)\}\{x-(g+\beta)\}=0.$$

Expanding and letting $A=g^2-\alpha^2$ and $B=g^2-\beta^2$, gives

$$x^4+(-4g^2+A+B)x^2+(2g)(B-A)x+AB=0.$$

[3]For a 3D version of Euler's solution in which the $\pm g_i$ are associated with the mid-points of the six edges of a regular tetrahedron, see Fig 2 in Nickalls (2012), The quartic equation: alignment with an equivalent tetrahedron, *Mathematical Gazette*, 96, 49–55; http://www.nickalls.org/dick/papers/maths/tetrahedron2012.pdf.

We can eliminate α, β by first equating coefficients with the monic form of Equation 1 giving

$$\begin{cases} p/a &= -4g^2 + A + B, \\ q/a &= 2g(B - A), \\ r/a &= AB, \end{cases}$$

and then eliminating A and B (using the identity $4AB = 2A \times 2B$), which generates a resolvent sextic in g, the roots of which are the six values $\pm g_1, \pm g_2, \pm g_3$. The substitution $g^2 \mapsto x$ then generates Euler's original resolvent cubic [1, 2, 3, 4]

$$R(x) \equiv x^3 + \frac{p}{2a}x^2 + \left(\frac{p^2 - 4ar}{16a^2}\right)x - \frac{q^2}{64a^2} = 0, \tag{4}$$

whose roots r_i are therefore $g_1^2,\ g_2^2,\ g_3^2$. The four roots of the reduced quartic $Z(x) = 0$ are among the eight possible values of $\pm\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3}$; but in order to determine which four they are we need a way of allocating the signs correctly.

Euler, using a monic quartic of the form $x^4 - lx^2 - mx - n = 0$, says he resolved the sign problem by noting that $\sqrt{r_1 r_2 r_3} = m/8$, as follows [3, § 773]:

> ... But it is to be observed, that the product ... $\sqrt{r_1 r_2 r_3}$, must be equal to $m/8$, and that if $m/8$ be positive, the product of the terms $\sqrt{r_1}$, $\sqrt{r_2}$, $\sqrt{r_3}$, must likewise be positive;

Unfortunately Euler did not elaborate further on this, but the key to understanding the sign problem is not difficult to find, since from Equation 2 we have

$$\begin{aligned} 8g_1g_2g_3 &= (z_3 + z_4)(z_3 + z_1)(z_3 + z_2), \\ &= z_3^3 + z_3^2(z_1 + z_2 + z_4) + z_3(z_2z_1 + z_2z_4 + z_1z_4) + z_4z_1z_2. \end{aligned}$$

Now $z_1 + z_2 + z_4 = -z_3$ (since $\Sigma z_j = 0$), hence

$$8g_1g_2g_3 = z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_1 + z_4z_1z_2, \tag{5}$$

and so $8g_1g_2g_3$ is actually one of the four elementary symmetric functions of the roots z_j. Its value is therefore equal to $-1\times$ the coefficient of the x-term of the monic form of the reduced quartic equation $Z(x) = 0$, and so we have

$$8\sqrt{r_1 r_2 r_3} = 8g_1g_2g_3 = -q/a, \tag{5a}$$

which is equivalent to Euler's $\sqrt{r_1 r_2 r_3} = m/8$.

4 Geometric basis for the sign of $\sqrt{r_1 r_2 r_3}$

A useful way of 'seeing' the quartic algebra at work is to express the coefficients in terms of the key 'visible' parameters ε, y_{N_z}, $y_{N_z'}$ shown in Figure 2, as follows: Let $F(X)$ be a quartic polynomial with real coefficients ($a \neq 0$)

$$F(X) \equiv aX^4 + bX^3 + cX^2 + dX + e, \tag{6}$$

with invariants [6, p. 76]

$$\begin{cases} G &= b^3 + 8a^2d - 4abc, \\ H &= 8ac - 3b^2, \\ I &= 12ae - 3bd + c^2, \\ J &= 72ace + 9bcd - 27ad^2 - 27eb^2 - 2c^3. \end{cases} \tag{7}$$

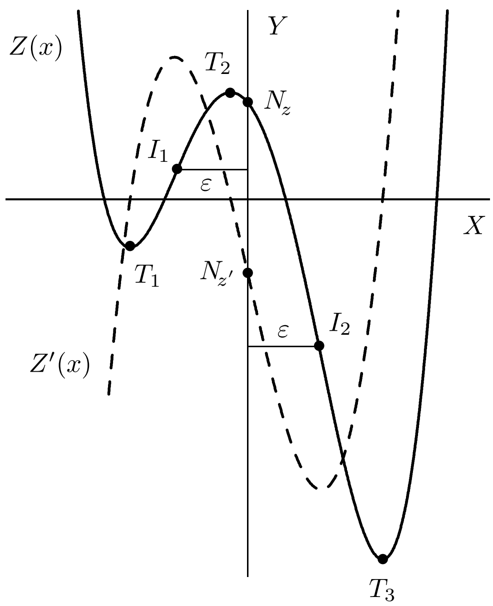

Figure 2:
The reduced quartic $Z(x)$, turning points (T_1, T_2, T_3), points of inflection (I_1, I_2), and first differential $Z'(x)$. The x-coordinates of the points of inflection are $\pm\varepsilon$. The curves intersect the y-axis at points N_z and $N_{z'}$.

Let its reduced form $Z(x)$ be generated by the translation $X \mapsto x + X_{Nf}$, where $X_{Nf} = -b/(4a)$. Using Taylor's theorem we have

$$Z(x) \equiv F(x + X_{Nf}) = ax^4 + \frac{F''(X_{Nf})}{2}x^2 + F'(X_{Nf})x + F(X_{Nf}). \tag{8}$$

If $Z(x)$ and $Z'(x)$ intersect the y-axis in points N_z and $N_{z'}$ respectively, then Equation 8 can be expressed as

$$Z(x) \equiv ax^4 - 6a\varepsilon^2 x^2 + y_{N_{z'}}x + y_{N_z} \tag{9}$$

where (see Equation 4 and Figures 2, 3)

$$\left\{ \begin{aligned} \varepsilon^2 &= \frac{(3b^2 - 8ac)}{48a^2} \equiv \frac{-H}{48a^2} \equiv \frac{-p}{6a}, \\ y_{N_z} &= F(X_{Nf}) \equiv \frac{I}{12a} - \frac{3H^2}{48^2 a^3} \equiv r, \\ y_{N_{z'}} &= F'(X_{Nf}) \equiv \frac{G}{8a^2} \equiv q, \\ -12a\varepsilon^2 &= F''(X_{Nf}). \end{aligned} \right. \tag{10}$$

Expressing the reduced quartic $Z(x)$ in this form (Equation 9) greatly facilitates visualisation, since we can now 'see' how the configuration of the curves $Z(x)$ and $Z'(x)$ is related to the coefficients. For example (assuming $a > 0$), if the x^2 term is positive then ε is complex ($\varepsilon^2 < 0$), and so the quartic will have two complex points of inflection and hence only one real turning point (cf. [10]).

If x_{T_i} are the x-coordinates of the turning points of $Z(x)$, then by differentiating Equation 9 we have (see Equations 5a and 10)

$$4x_{T_1}x_{T_2}x_{T_3} = \frac{-y_{N_{z'}}}{a} = 8\sqrt{r_1r_2r_3}, \tag{11}$$

and hence the sign of $\sqrt{r_1r_2r_3}$ is the same as that of $-y_{N_{z'}}/a$ and $x_{T_1}x_{T_2}x_{T_3}$. It follows, therefore, that we can actually 'see' the correct sign of $\sqrt{r_1r_2r_3}$ simply by observing the signs of the abscissae of the turning points of the reduced quartic, or by noting the location of $N_{z'}$ in relation to the abscissa.

For example (assuming $a > 0$), if the roots z_j are such that the middle turning point, T_2, is to the left of the y-axis, then not only will $y_{N_{z'}}$ be negative (Figure 2) but just two of the three x_{T_i} will be negative resulting in a positive product for $x_{T_1}x_{T_2}x_{T_3}$, and hence $\sqrt{r_1r_2r_3}$ will also be positive (see Equation 11). Conversely, if the middle turning point is to the right of the y-axis, then $y_{N_{z'}}$ will be positive, and only one of the x_{T_i} will be negative making the product $x_{T_1}x_{T_2}x_{T_3}$ negative.

5 Roots

As regards the roots z_j of the reduced quartic $Z(x)$, we can initially choose *any* sign combination for the $\sqrt{r_i}$, and then evaluate the sign of the product $\sqrt{r_1r_2r_3}$. If the sign of the product is the *same* as that of $-y_{N_{z'}}/a$ (see Equation 11) then we have a valid combination of signs, and can proceed to determine the four z_j using Equation 3. Otherwise, it is only necessary to change the sign of any *one* of the $\sqrt{r_i}$ (say, $\sqrt{r_1} \to -\sqrt{r_1}$), and proceed as before using Equation 3.

When the reduced quartic is symmetric about the y-axis one of the x_{T_i} will be zero and hence the product $\sqrt{r_1r_2r_3}$ is zero. However, the solution in this case is trivial since $Z(x)$ is then an even function as $y_{N_{z'}}$ is also zero.

6 Application

Since all resolvent cubics of the quartic can be transformed to a standard form [9], typically expressed as [6, p. 77]

$$T(x) \equiv x^3 - 3Ix + J, \tag{12}$$

we can solve any quartic by solving instead a simple reduced form of the resolvent, say $T(x) = 0$, and then recover the roots of Euler's resolvent using the transformation which carries the reduced form back to $R(x)$.

For example, the translation $x \mapsto x + x_{N_r}$ to reduce $R(x)$, for which $x_{N_r} = -p/(6a) \equiv \varepsilon^2$, generates the reduced form $S(x)$, as follows:

$$S(x) \equiv R(x+\varepsilon^2) \equiv x^3 - \frac{I}{48a^2}x + \frac{J}{1728a^3}. \tag{13}$$

The substitution $x \mapsto x/(12a)$ then scales $1728a^3S(x)$ to $T(x)$, and hence if the roots of $S(x) = 0$ and $T(x) = 0$ are s_i and t_i respectively, then

$$r_i = s_i + \varepsilon^2 = \frac{t_i}{12a} + \varepsilon^2. \tag{14}$$

This convenient approach is illustrated in Example 1.

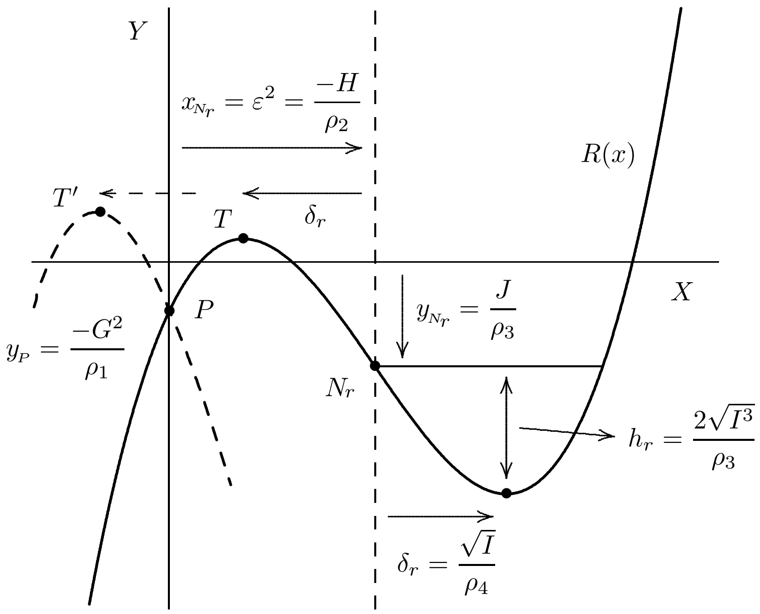

Figure 3:
Euler's resolvent cubic $R(x)$ with three real roots ($h_r^2 > y_{N_r}^2$, i.e. $4I^3 > J^2$) which are all positive ($\varepsilon^2 > 0$, $x_{N_r}^2 > \delta_r^2$). The conditions $\varepsilon^2 > 0$, $x_{N_r}^2 < \delta_r^2$ are associated with two negative roots (dashed curve). Note that G^2, H, I, J are constant multiples respectively of the resolvent's geometric parameters y_P, x_{N_r}, δ_r^2, y_{N_r} ($\rho_1 = 64^2a^6$, $\rho_2 = 48a^2$, $\rho_3 = 1728a^3$, $\rho_4 = 12a$).

The invariants I, J are readily visualised since any reduced cubic can be expressed in terms of its geometric parameters δ and y_N as in [8]

$$Ax^3 - 3A\delta^2 x + y_N = 0. \tag{15}$$

For example, equating coefficients between $S(x)$, $T(x)$ and the monic form of Equation 15, and noting that $h^2 = 4A^2\delta^6$ [8], shows that I, J are simply constant multiples of δ^2, y_N as follows (Figure 3):

$$\begin{cases} A_r = A_s = A_t = 1, \\ I = \delta_r^2(12a)^2 = \delta_s^2(12a)^2 = \delta_t^2, \\ J = y_{N_r}(12a)^3 = y_{N_s}(12a)^3 = y_{Nt}, \\ \dfrac{4I^3}{J^2} = \left(\dfrac{h_r}{y_{N_r}}\right)^2 = \left(\dfrac{h_s}{y_{N_s}}\right)^2 = \left(\dfrac{h_t}{y_{Nt}}\right)^2. \end{cases} \tag{16}$$

Thus each of these invariants has a visible geometric interpretation in relation to Euler's resolvent cubic, either as a position parameter with respect to the axes (G, H, J), or as a shape parameter (I). For example, we can now see that the condition $J = 0$ simply indicates that the N-point of the resolvent cubic lies on the x-axis and all that that implies (see Example 2). Similarly, the condition $I = 0$ indicates that the resolvent adopts the 'cubic parabola' form. Furthermore $y_P \leq 0$, which reveals how and why the resolvent cubic cannot have just a single negative root [4]. The syzygy $-27G^2 = H^3 - 48a^2IH + 64a^3J$ [6, p. 76] is generated by substituting into $S(x)$ the coordinates of P $(H/(48a^2), -G^2/(64^2a^6))$.

[4]For real coefficients $G^2 \geq 0$, and hence P must always be on or below the x-axis.

7 Euler's cubic and the quartic root configurations

A very significant but seemingly overlooked aspect of Euler's resolvent cubic is its beautiful and symmetric relationship with two important algebraic objects, namely the discriminant $4I^3 - J^2$ and the seminvariant $H^2 - 16a^2I$, the signs of which distinguish between the various quartic root configurations [5, § 68; 6, p. 80; 7, p. 28]. Visualising the resolvent in relation to the invariants (Figure 3) reveals the mechanisms, as follows:

7.1 $4I^3 - J^2$

Since $h^2 = 4A^2\delta^6$ [8], it follows from Equation 16 that

$$\frac{-(4I^3 - J^2)}{12^6 a^6} = y_{N_r}^2 - h_r^2. \tag{17}$$

Thus the quartic discriminant $4I^3 - J^2$ is simply a constant multiple of $y_{N_r}^2 - h_r^2$, the sign of which reflects whether the x-axis lies *between* ($y_{N_r}^2 < h_r^2$), *on* ($y_{N_r}^2 = h_r^2$), or *outside* ($y_{N_r}^2 > h_r^2$) the turning points of the resolvent cubic (Figure 3).

7.2 $H^2 - 16a^2I$

The sign of this algebraic object distinguishes (when $\varepsilon^2 > 0$) between the then two possible quartic root configurations associated with the case $4I^3 - J^2 > 0$, namely (a) four real roots ($H^2 - 16a^2I > 0$), and (b) four complex roots ($H^2 - 16a^2I < 0$) [5, § 68]. Substituting for H (Equation 10) and I (Equation 16) gives

$$H^2 - 16a^2I = (-48a^2\varepsilon^2)^2 - 16a^2(12^2a^2\delta_r^2) = 3^2 4^4 a^4(\varepsilon^4 - \delta_r^2).$$

But $\varepsilon^2 = x_{N_r}$ (Figure 3) and hence

$$\frac{H^2 - 16a^2I}{3^2 4^4 a^4} = x_{N_r}^2 - \delta_r^2. \tag{18}$$

Thus $H^2 - 16a^2I$ is just a constant multiple of $x_{N_r}^2 - \delta_r^2$, the sign of which (when $\varepsilon^2 > 0$) reflects whether the y-axis lies *between* ($x_{N_r}^2 < \delta_r^2$), *on* ($x_{N_r}^2 = \delta_r^2$), or *outside* ($x_{N_r}^2 > \delta_r^2$) the turning points of the resolvent cubic (cf. [6, p. 80, proposition 7]).

For example (Figure 3), when a quartic with three real turning points ($\varepsilon^2 > 0$) has four real roots ($4I^3 - J^2 > 0$) Euler's cubic $R(x)$ has three positive real roots—the y-axis lies *outside* the two turning points—and so $x_{N_r}^2 > \delta_r^2$ and hence $H^2 - 16a^2I > 0$.

Conversely, when a quartic with three real turning points ($\varepsilon^2 > 0$) has four complex roots ($4I^3 - J^2 > 0$), $R(x)$ then has exactly two negative real roots, and so its turning point T' (Figure 3) lies to the left of the y-axis ($x_{N_r}^2 < \delta_r^2$), hence $H^2 - 16a^2I < 0$.

7.3 A new class of object?

Since $H^2 - 16a^2I$ functions with regard to the y-axis in *exactly* the same way that $4I^3 - J^2$ functions with regard to the x-axis, I would like to suggest that this pair of algebraic objects should be regarded as forming a distinct class of object—thereby linking two previously independent algebraic quantities with a single unifying concept.

8 Example 1

Solve $f(X) \equiv X^4 - 11X^3 + 41X^2 - 61X + 30 = 0$.

The key parameters are: $a = 1$, $X_{N_f} = 11/4$, $Y_{N_{f'}} = f'(X_{N_f}) = -15/8$, $G = -15$, $I = 28$, $J = -160$, $\varepsilon^2 = 35/48$. Using say, $T(x)$, we solve [5]

$$T(x) \equiv x^3 - 84x - 160 = 0,$$

the three t_i being $-8, -2, 10$. The $\sqrt{r_i}$ are therefore given by

$$\begin{cases} \sqrt{r_1} = \sqrt{\varepsilon^2 + \dfrac{t_1}{12a}} = \sqrt{\dfrac{35}{48} - \dfrac{8}{12}} = \dfrac{1}{4}, \\ \sqrt{r_2} = \sqrt{\varepsilon^2 + \dfrac{t_2}{12a}} = \sqrt{\dfrac{35}{48} - \dfrac{2}{12}} = \dfrac{3}{4}, \\ \sqrt{r_3} = \sqrt{\varepsilon^2 + \dfrac{t_3}{12a}} = \sqrt{\dfrac{35}{48} + \dfrac{10}{12}} = \dfrac{5}{4}. \end{cases}$$

Since the sign of $-Y_{N_{f'}}/a$ is positive [6] then the product of the $\sqrt{r_i}$ must also be positive—which it is. Finally, adding X_{N_f} recovers the quartic roots ($X_j = X_{N_f} \pm \sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3}$) using (3) as follows:

$$\begin{cases} X_1 = \dfrac{11}{4} - \dfrac{1}{4} + \dfrac{3}{4} - \dfrac{5}{4} = 2, \\ X_2 = \dfrac{11}{4} - \dfrac{1}{4} - \dfrac{3}{4} + \dfrac{5}{4} = 3, \\ X_3 = \dfrac{11}{4} + \dfrac{1}{4} + \dfrac{3}{4} + \dfrac{5}{4} = 5, \\ X_4 = \dfrac{11}{4} + \dfrac{1}{4} - \dfrac{3}{4} - \dfrac{5}{4} = 1. \end{cases}$$

Even the solution of $T(x) = 0$ is greatly simplified since δ, h, y_N are simple functions of I and J (see Equation 16). For example, $T(x) = 0$ has three real roots in this case since $(y_{N_t}/h_t)^2 \equiv J^2/(4I^3) \le 1$ [8].

9 Example 2

Explain the significance of $J = 0$, $I > 0$, *for a quartic with four real roots.*

The condition $J = 0$ implies that Euler's resolvent cubic has its N-point on the x-axis (Figure 3), and hence it has three roots in arithmetic progression. If also $I > 0$ (resolvent cubic has two real turning points), then the resolvent's roots are distinct and (with the root at infinity) form a harmonic range. Since the roots of the parent quartic have the same cross-ratio they also form a harmonic range.

[5] Note that we could instead solve $S(x) = 0$, and then use $r_i = \varepsilon^2 + s_i$ (see Equation 14).
[6] Since $Y_{N_{f'}} \equiv G/(8a^2)$ it is probably more convenient to use the sign of $-G/a$ instead.

10 Acknowledgements

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