A new algorithm for generating Pythagorean triples [1]

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1 Introduction

The following observation by one of us (RWDN) arose while investigating Pythagorean triples (x, y, z) with $y = x+1$, and led to some interesting relationships which allow Pythagorean triples to be generated iteratively. In this article we obtain these and other algorithms which, as far as we are aware, have not been described before. In addition, we show that some related relationships, recently described by Hatch (1995) and by Mills (1996), are a consequence of these algorithms.

Let $z = y+b$. So $x^2+y^2 = (y+b)^2$, and hence $y = (x^2-b^2)/2b$. The triple can therefore be expressed as

$$x,\ \left(\frac{x^2-b^2}{2b}\right),\ \left(\frac{x^2+b^2}{2b}\right).$$

Let $y = x+a$. So $x+a = (x^2-b^2)/(2b)$ and thus

$$x^2-2bx-(b^2+2ab) = 0. \tag{1}$$

2 Triples with $y = x+1$

In this case we have $a = 1$, and equation (1) becomes

$$x^2-2bx-(b^2+2b) = 0. \tag{2}$$

If, also, $b = 1$ then equation (2) becomes

$$x^2-2x-3 = (x-3)(x+1) = 0.$$

It follows that the two roots 3, -1 of this quadratic generate the triples $3,4,5$ and $-1,0,1$ respectively.

[1] This minor revision fixes typos, adds a footnote in § 7, and also includes some URLs for the references. The original published version is available from http://www.jstor.org/stable/3620161

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In order to generate the next triple in the $a = 1$ sequence (i.e., $y = x + 1$) using equation (2) it is necessary to know the next value of b. Now the next triple in the $a = 1$ sequence is $20, 21, 29$, for which $b = 8$. Since $8 = 3 + 5$ the relationship $b_{n+1} = X_n + Z_n$ suggested itself as a possibility to be explored for determining subsequent values of b, where X_n, Y_n, Z_n represent the n^{th} Pythagorean triple of a sequence. Using this notation, $b_2 = 3 + 5 = 8$, and equation (2) becomes

$$x^2 - 16x - 80 = (x - 20)(x + 4) = 0,$$

which generates the second $a = 1$ triple $20, 21, 29$. The first four triples in the $a = 1$ sequence can be generated using equation (2) as follows:

$$\begin{array}{ccccc} b_1 = 1 & \Rightarrow & x^2 - 2x - 3 = 0 = (x-3)(x+1) & \Rightarrow & 3,\ 4,\ 5 \\ b_2 = 3+5 & \Rightarrow & x^2 - 16x - 80 = 0 = (x-20)(x+4) & \Rightarrow & 20,\ 21,\ 29 \\ b_3 = 20+29 & \Rightarrow & x^2 - 98x - 2499 = 0 = (x-119)(x+21) & \Rightarrow & 119,\ 120,\ 169 \\ b_4 = 119+169 & \Rightarrow & x^2 - 576x - 83520 = 0 = (x-696)(x+120) & \Rightarrow & 696,\ 697,\ 985 \end{array}$$

When the working is laid out like this a clear pattern emerges which facilitates factorisation, namely that for the case where $y = x + 1$ then $(x + Y_{n-1})$ is a factor, and equation (2) factorises as follows:

$$x^2 - 2b_n x - \left(b_n{}^2 + 2b_n\right) = (x - X_n)(x + Y_{n-1}) = 0.$$

In view of these interesting heuristic relationships a more systematic approach was explored which led to a solution via Pell's equation (see Davenport, 1970), which we now present.

3 Triples where a is constant

Consider the triple X, Y, Z where $Y = X + a \quad (a > 0)$. Then

$$X^2 + (X + a)^2 = Z^2.$$

Expanding and doubling both sides gives

$$4X^2 + 4aX + 2a^2 = 2Z^2,$$

$$\text{i.e.,} \qquad (2X + a)^2 + a^2 = 2Z^2,$$

$$\text{i.e.,} \qquad \left(\frac{2X + a}{a}\right)^2 + 1 = 2\left(\frac{Z}{a}\right)^2.$$

Write $(2X + a)/a = U$ and $Z/a = T$. Then

$$U^2 + 1 = 2T^2.$$

If $a = 1$ then we have an equation in integers ($U, T \in \mathbb{Z}$), and therefore this is a case of Pell's equation for which it is known that the n^{th} solution U_n, T_n in positive integers is given by

$$U_n + T_n\sqrt{2} = (1 + \sqrt{2})^{2n-1}.$$

Thus

$$\begin{aligned} U_{n+1} + T_{n+1}\sqrt{2} &= (1+\sqrt{2})^{2(n+1)-1}, \\ &= (1+\sqrt{2})^2(1+\sqrt{2})^{2n-1}, \\ &= (1+\sqrt{2})^2(U_n + T_n\sqrt{2}), \\ &= (3+2\sqrt{2})(U_n + T_n\sqrt{2}), \\ &= (3U_n + 4T_n) + (2U_n + 3T_n)\sqrt{2}. \end{aligned}$$

Equating $\sqrt{2}$ and non-$\sqrt{2}$ parts gives

$$T_{n+1} = 2U_n + 3T_n, \tag{3}$$
$$U_{n+1} = 3U_n + 4T_n. \tag{4}$$

Although these recurrence relations have been reached using Pell's equation for integers, we wish to highlight the following observation. Suppose that U_n, T_n are determined recursively by equations (3) and (4), whether or not they are integers. Then direct calculation gives

$$\begin{aligned} 2T_{n+1}^2 - U_{n+1}^2 &= 2(2U_n + 3T_n)^2 - (3U_n + 4T_n)^2, \\ &= 2{T_n}^2 - {U_n}^2. \end{aligned}$$

Thus, provided $2{T_1}^2 - {U_1}^2 = 1$, the pairs U_n, T_n are solutions of $2T^2 - U^2 = 1$, whether or not U_1, T_1 are integers.

Substituting for T and U with $T = Z/a$ and $U = (2X+a)/a$ in equations (3) and (4) (bearing in mind that the full set of Pythagorean triple solutions is given only for the case where $a = 1$) gives respectively

$$\left(\frac{Z_{n+1}}{a}\right) = 2\left(\frac{2X_n + a}{a}\right) + \left(\frac{3Z_n}{a}\right),$$

and

$$\left(\frac{2X_{n+1} + a}{a}\right) = 3\left(\frac{2X_n + a}{a}\right) + \left(\frac{4Z_n}{a}\right),$$

which reduce to

$$Z_{n+1} = 4X_n + 3Z_n + 2a, \tag{5}$$
$$X_{n+1} = 3X_n + 2Z_n + a. \tag{6}$$

Since $Y_n = X_n + a$ we can rearrange equation (6) to obtain

$$X_{n+1} = 2(X_n + Z_n) + Y_n. \tag{7}$$

Subtracting equation (6) from equation (5) gives

$$\begin{aligned} Z_{n+1} - X_{n+1} &= Z_n + X_n + a, \\ \text{i.e.,} \qquad b_{n+1} + a &= Z_n + X_n + a, \\ \text{i.e.,} \qquad b_{n+1} &= Z_n + X_n, \end{aligned} \tag{8}$$

the relationship conjectured above in Section 2.

4 Triples where b is constant

Similar relationships can be obtained for the case where b is constant, as follows:

$$a_{n+1} = X_n + Y_n, \tag{9}$$
$$X_{n+1} = X_n + 2b, \tag{10}$$
$$Y_{n+1} = 2X_n + Y_n + 2b, \tag{11}$$
$$Z_{n+1} = 2X_n + Z_n + 2b. \tag{12}$$

We leave it to the interested reader to verify that these do give a sequence of triples, and that in this case equation 1 factorises as follows:

$$x^2 - 2bx - (b^2 + 2ba_n) = (x - X_n)(x + X_{n-1}) = 0.$$

5 Conclusion

We have therefore shown that the following relationships generate Pythagorean triples X, Y, Z

$$\text{For } a = \text{constant} \quad (X_n + a = Y_n) \quad \begin{cases} X_{n+1} = 3X_n + 2Z_n + a, \\ b_{n+1} = X_n + Z_n. \end{cases}$$

$$\text{For } b = \text{constant} \quad (Z_n = Y_n + b) \quad \begin{cases} Z_{n+1} = 2X_n + Z_n + 2b, \\ a_{n+1} = X_n + Y_n. \end{cases}$$

In the special cases $a = 1$ ($a =$ constant) and $b = 1$ ($b =$ constant) then the above recurrence relationships generate *all* the relevant corresponding Pythagorean triples. However, for all other cases the above relationships necessarily give only some of the triples, since the recurrence relations (equations (3) and (4)) do not, from any initial U_1, T_1 generate all fractional solutions of Pell's equation.

A significant feature of these algorithms is that if the initial triple is primitive, then *every* member of the generated sequence is also primitive. For example, consider the $a =$ constant case. Suppose $h|X_{n+1}$ and $h|Y_{n+1}$, then $h^2|(X_{n+1}^2 + Y_{n+1}^2)$, and hence $h|Z_{n+1}$. Also, $h|(Y_{n+1} - X_{n+1})$, and so $h|a$. From equations 5 and 6 it follows that $h|(4X_n + 3Z_n)$ and $h|(3X_n + 2Z_n)$; by taking linear combinations of these we see that $h|Z_n$ and $h|X_n$, from which it follows that $h|Y_n$, as $Y_n = X_n + a$. It follows, therefore, that if the triple X_{n+1}, Y_{n+1}, Z_{n+1} is not primitive, then neither is X_n, Y_n, Z_n; and so on down to X_1, Y_1, Z_1. A similar argument for the $b =$ constant case applies with equations (10), (11), (12).

6 Examples

6.1 Constant a, ($a = 1$)

Suppose we start with the primitive Pythagorean triple $119, 120, 169$ and wish to generate subsequent triples in the sequence. Using equation (6) we obtain

$$X_2 = 3X_1 + 2Z_1 + a = 3(119) + 2(169) + 1 = 696,$$

and so the next triple is $696, 697, 985$.

6.2 Constant a, ($a = 7$)

Start with the primitive Pythagorean triple $5, 12, 13$ ($a = 7$). In this case our algorithm generates $X_2 = 48$, and hence the triple $48, 55, 73$, which misses the triple $21, 28, 35$.

6.3 Constant b, ($b = 2$)

Start with the primitive Pythagorean triple $8, 15, 17$. Using equation (10) we obtain

$$Z_2 = 2X_1 + Z_1 + 2b = 2(8) + 17 + 2(2) = 37,$$

and so our next triple is $12, 35, 37$, which misses the triple $10, 24, 26$.

7 Alternative relationships for X_{n+1} and Z_{n+1}

The empirical relationship $X_{n+1} = 6X_n - X_{n-1} + 2$, described in a recent article by Hatch (1995) [4]. for the case when $a = 1$, is in fact a particular case of a more general relationship which follows directly from our approach, as follows.
From equation (6)

$$X_n = 3X_{n-1} + 2Z_{n-1} + a = 2(X_{n-1} + Z_{n-1}) + X_{n-1} + a. \tag{13}$$

From equation (8)

$$b_n = X_{n-1} + Z_{n-1}.$$

Eliminating $(X_{n-1} + Z_{n-1})$ between equations (13) and (14) gives

$$X_n = 2b_n + X_{n-1} + a.$$

But $X_n + a + b_n = Z_n$, so

$$X_n = (2Z_n - 2X_n - 2a) + X_{n-1} + a = 2Z_n - 2X_n + X_{n-1} - a.$$

Eliminating $(2Z_n)$ using equation (6) gives

$$X_n = (X_{n+1} - 3X_n - a) - 2X_n + X_{n-1} - a,$$

$$\text{i.e.} \qquad X_{n+1} = 6X_n - X_{n-1} + 2a. \tag{14}$$

When $a = 1$ then this is Hatch's equation

$$X_{n+1} = 6X_n - X_{n-1} + 2,$$

as required.

Finally, a similar argument shows that the relationship $Z_{n+1} = 6Z_n - Z_{n-1}$ discussed by Mills (1996) for a sequence with a constant, also follows from our approach.
From equation (5)

$$2Z_{n+1} = 8X_n + 6Z_n + 4a. \tag{15}$$

From equation (6)

$$3X_{n+1} = 9X_n + 6Z_n + 3a. \tag{16}$$

[4] We are indebted to J. Pla for drawing our attention to the fact that the Hatch equation was also the subject of the following article: Osborne GA (1914). A problem in number theory. *American Mathematical Monthly; vol 21 (No. 5, May), pp. 148–150; http://www.jstor.org/stable/2972176*

Eliminating Z_n between equations (16) and (17) gives

$$2Z_{n+1} = 3X_{n+1} - X_n + a.$$

Using the corresponding equations for Z_n and Z_{n-1} we see that

$$2\left(Z_{n+1} - 6Z_n + Z_{n-1}\right) = 3\left(X_{n+1} - 6X_n + X_{n-1}\right) - \left(X_n - 6X_{n-1} + X_{n-2}\right) - 4a,$$

which, by equation (15), gives

$$2\left(Z_{n+1} - 6Z_n + Z_{n-1}\right) = 3(2a) - 2a - 4a = 0.$$

Thus

$$Z_{n+1} = 6Z_n - Z_{n-1},$$

as required.

8 References

- Davenport H. (1970). *The higher arithmetic; an introduction to the theory of numbers.* (Cambridge University Press).
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- Mills J.T.S. (1996). Another family tree for Pythagorean triples. *Mathematical Gazette*; 80 (November), pp. 545–548. (http://www.jstor.org/stable/3618521)

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