

# A line-and-conic theorem having a visual correlate<sup>1</sup>

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<http://www.nickalls.org/dick/papers/math/lineandconic1986.pdf>

## 1 Introduction

This article describes a new and general line-and-conic property<sup>3</sup> leading to a consideration of a ‘six-point’ circle and the related geometry of an interesting visual illusion.

**THEOREM:** *The sum of the two angles included by the tangents to any conic from the ends of any straight line, is equal to the sum of the angles subtended by the line at the foci.*

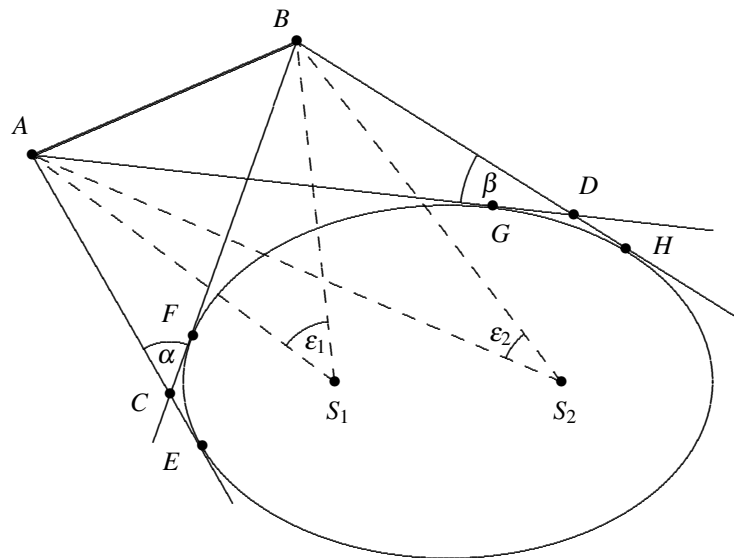


Figure 1:  $\alpha + \beta = \epsilon_1 + \epsilon_2$

<sup>1</sup>This minor revision fixes some typographic errors and incorporates some headings, explanatory footnotes, and minor typographic improvements to Figure 1 & Figure 2. The original published version is available from <http://www.jstor.org/stable/3615821>

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<sup>3</sup>This theorem was later generalised:– Nickalls RWD (2000). A conic theorem generalised: directed angles and applications. *The Mathematical Gazette*, 84 (July), 232–241 (<http://www.nickalls.org/dick/papers/math/conicthm2000.pdf>).

TO PROVE:  $\alpha + \beta = \varepsilon_1 + \varepsilon_2$  (see Figure 1).

As  $CS_1$  and  $DS_1$  bisect angles  $ES_1F$  and  $GS_1H$  respectively, let  $ES_1F = 2\alpha_1$ , and  $GS_1H = 2\beta_1$ . Then, as  $AS_1$  and  $BS_1$  bisect angles  $ES_1G$  and  $FS_1H$  respectively, it follows fairly easily that

$$\varepsilon_1 = \alpha_1 + \beta_1,$$

and, with similar labelling for  $S_2$ ,  $\varepsilon_2 = \alpha_2 + \beta_2$ . Therefore

$$\varepsilon_1 + \varepsilon_2 = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2).$$

By considering the triangle pairs  $CFS_1$ ,  $CFS_2$  and  $DGS_1$ ,  $DGS_2$ , it can be shown that

$$\alpha = \alpha_1 + \alpha_2 \text{ and } \beta = \beta_1 + \beta_2,$$

from which it follows, as required, that

$$\varepsilon_1 + \varepsilon_2 = \alpha + \beta. \tag{1}$$

(Although the sign of the angles in equation 1 clearly varies according to the orientation of line and conic, a simple sign convention<sup>4</sup> allows the relative signs to be readily determined.)

In the case of the parabola, as one of the foci is at infinity, one of the angles at the foci must be zero. The equation therefore reduces to  $\varepsilon = \alpha + \beta$ . In the case of the circle, let  $\varepsilon_1 = \varepsilon_2 = \varepsilon$ . The equation then reduces to  $2\varepsilon = \alpha + \beta$ .

## 2 Six-point circle

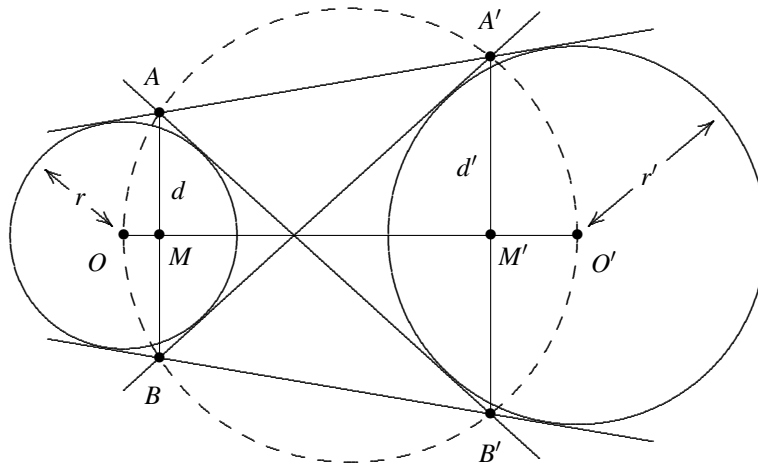


Figure 2:

<sup>4</sup>Nickalls (2000). See footnote 3 for URL.

Now applying this theorem to two circles facilitates the following ‘six-point’ circle proof (see Figure 2). Let  $O$  and  $O'$  be the centres of circles whose respective radii are  $r$  and  $r'$ . Let the internal and external tangents intersect in  $A, A', B, B'$ . As  $AB$  and  $A'B'$  are perpendicular to  $OO'$ , then  $AA'B = AB'B$ . It follows from the above result that  $AA'B + AB'B = 2AO'B$  and therefore  $AA'B = AB'B = AO'B$ . Similarly  $A'AB' = A'BB' = A'OB'$  and so the two centres  $O$  and  $O'$  lie on the circle  $A, A', B, B'$ . Therefore the six points  $A, A', B, B', O, O'$  are concyclic. Furthermore, if  $AB = 2d$  and  $A'B' = 2d'$ , it follows that  $rr' = dd'$ . The details of the proof of this result are left to the interested reader.

### 3 Locus property

Finally, the following locus property which has an interesting visual correlate, follows directly from the above results. Consider the circle  $ABO$  in Figure 3, where  $M$  is the midpoint of  $AB$ , and  $AB \perp MO$ .

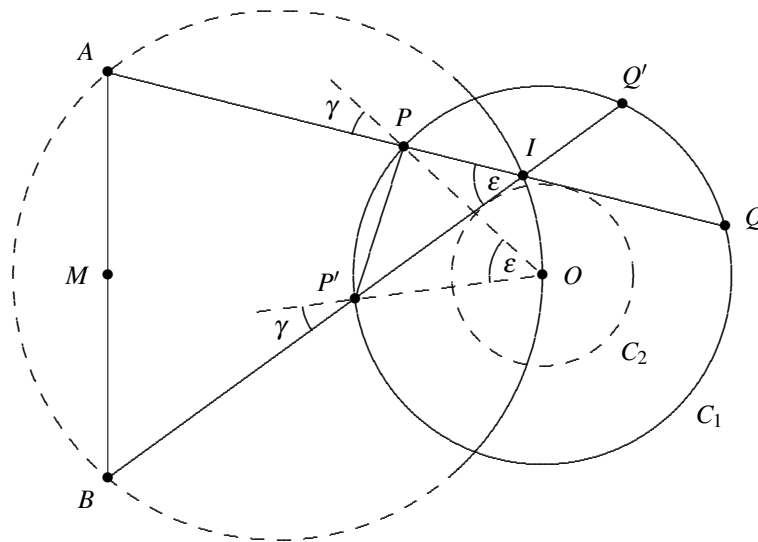


Figure 3:

**THEOREM.** *If  $PP'$  is a variable chord of any circle ( $C_1$ ) subtending the same constant angle at the centre ( $O$ ) as that subtended by  $AB$  at  $O$ , then the locus ( $I$ ) of the intersection of  $AP$  and  $BP'$  is an arc of the circle  $ABO$ .*

**PROOF.** Construct any circle ( $C_2$ ) smaller than and concentric with  $C_1$  and let tangents from  $A$  and  $B$  to the circle  $C_2$  intersect in  $I$  and cut  $C_1$  in  $P$  and  $P'$ , as shown in Figure 3. As an easy consequence of the earlier result, angles  $AIB$  and  $AOB$  are equal and so  $I$  lies on the circle  $ABO$ . By the same argument,  $POP' = PIP'$ , and therefore  $POP' = AOB$ . From this it follows that  $QOQ' = POP'$ , and therefore the intersection  $I$  lies on the circle

$ABO$  (irrespective of whether the chord is at  $PP'$  or  $QQ'$ ). This holds for all circles  $C_2$ , and so the locus  $I$  is an arc of the circle  $ABO$  for all positions of the chord  $PP'$ .

Furthermore, since  $P, I, O, P'$  are concyclic (as  $PIP' = POP'$ ) it follows that  $IPO = IP'O$ . Thus the lines  $AP$  and  $BP'$  meet the circle  $C_1$  with the same angle of incidence  $\gamma$ .

Interestingly, the extent of the locus<sup>5</sup> depends on the position of the chord  $AB$  in relation to the circle  $C_1$ , being a simple arc if  $AB$  is outside  $C_1$ . However, if  $AB$  is a chord of  $C_1$ , then one revolution of  $PP'$  results in  $I$  making one complete revolution of  $ABO$ . If  $AB$  is inside  $C_1$ , then the locus is two revolutions of  $ABO$ .

## 4 A visual correlate (the Pulfrich effect)

Under certain circumstances it is possible to actually see the locus of the point  $I$  as the path of an object which appears to move back and forth along an arc of the circle  $ABO$ , in a striking visual illusion<sup>6</sup>.

Consider the eyes to be at the points  $A$  and  $B$ , viewing an object  $P$  rotating clockwise with constant angular velocity ( $\omega$ ) in a horizontal plane about a centre  $O$ . If  $P$  is viewed horizontally and binocularly with a neutral density filter in front of the right eye ( $B$ ), and with  $AB$  perpendicular to  $MO$ , then under certain conditions of viewing distance ( $MO$ ), angular velocity ( $\omega$ ), and filter density (i.e. when  $POP' = AOB$ ), the object  $P$  will seem not to be rotating about  $O$ , but will appear to be moving back and forth from side to side, passing through the centre  $O$  and describing an arc of the circle  $ABO$ .

The illusion is due to the fact that the filter introduces a slight visual time delay (say,  $t$ ) which is therefore equivalent to the angle  $POP' (= \omega t)$ . The filtered eye therefore, always sees the object where it was a few milliseconds ago (i.e. at  $P'$ ). It is thought that fusion of these two images results in the object  $P$  appearing to be at the position  $I$ .

This illusion is a particular manifestation of a little known phenomenon called the *Pulfrich effect*<sup>7</sup>, after Carl Pulfrich (1858–1927) who first investigated it at the Carl Zeiss laboratories in Jena. The geometry of this circular configuration gives an unusual way of considering this effect.



<sup>5</sup>The loci are detailed at: [http://pulfrich.siuc.edu/Pulfrich.Pages/lit\\_pulfr/nick\\_thm/1992\\_NickallsThm.htm](http://pulfrich.siuc.edu/Pulfrich.Pages/lit_pulfr/nick_thm/1992_NickallsThm.htm).

<sup>6</sup>A demo is available at <http://www.nickalls.org/dick/papers/pulfrich/rotatingdemo.pdf>

<sup>7</sup>Pulfrich, C. (1922). Die Stereoskopie im Dienste der isochromen und heterochromen Photometrie. *Naturwissenschaften*, 10, 553–564. ([http://pulfrich.siuc.edu/Pulfrich.Pages/lit\\_pulfr/1922\\_Pulfrich.htm](http://pulfrich.siuc.edu/Pulfrich.Pages/lit_pulfr/1922_Pulfrich.htm)).

See also:

—Nickalls RWD (1986). The rotating Pulfrich effect, and a new method of determining visual latency differences. *Vision Research*; 26, 367–372. (<http://www.nickalls.org/dick/papers/pulfrich/pulfrich1986.pdf>).

—Nickalls RWD (1996). The influence of target angular velocity on visual latency difference determined using the rotating Pulfrich effect. *Vision Research*; 36, 2865–2872. (<http://www.nickalls.org/dick/papers/pulfrich/pulfrich1996.pdf>).

—Nickalls RWD, Kazachkov AR, Vasylevska Yu.V and Kalinin VV (2002). Motional visual illusions on-line. *Proceedings of the 2002 International Conference on Information and Communication Technologies in Education (ICTE2002)*, Badajoz (Spain), November 13–16, 2002; p 1320–1324 (ISBN: Coleccion-84-95251-76-0). Available on-line in *Journal of Digital Contents*; vol 1(1), 2003; (<http://www.nickalls.org/dick/papers/pulfrich/spain2002pulfrich.pdf>).

—Nakamizo S, Nickalls RWD and Nawae H (2004). Visual latency difference determined by two ‘rotating’ Pulfrich techniques. *Swiss Journal of Psychology*; 63, 201–205. (<http://www.nickalls.org/dick/papers/pulfrich/pulfrich2004.pdf>).