Viète, Descartes and the cubic equation [1]

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http://www.nickalls.org/dick/papers/maths/descartes2006.pdf

The single biggest problem we face is that of visualisation
Richard P. Feynman (1918–1988) [1]

1 Introduction

An appreciation of the geometry underlying algebraic techniques invariably enhances understanding, and this is particularly true with regard to polynomials. With visualisation as our theme, this article considers the cubic equation and describes how the French mathematicians François Viète (1540–1603) and René Descartes (1596–1650) related the 'three-real-roots' case (*casus irreducibilis*) to circle geometry. In particular, attention is focused on a previously undescribed aspect, namely, how the lengths of the chords constructed by Viète and Descartes in this setting relate geometrically to the curve of the cubic itself.

In order to clarify the methods of Viète and Descartes we adopt the following strategies. Firstly, we relate the cubic equations of Viète and Descartes to the standard reduced form having geometric coefficients [2]

$$az^3 - 3a\delta^2 z + y_N = 0, \tag{1}$$

where $N(x_N, y_N)$ is the cubic's point of inflection, and $h = 2a\delta^3$ (see Figure 3). In view of the task in hand we will use the sine version of the standard 'three-real-root' solution (i.e. where $y_N/h = \sin 3\phi$) [2] as follows.

$$\left\{\begin{array}{l} z_1 = 2\delta \sin\phi, \\ z_2 = 2\delta \sin\left(2\pi/3 + \phi\right), \\ z_3 = 2\delta \sin\left(4\pi/3 + \phi\right). \end{array}\right. \tag{2}$$

Secondly, we use the calligraphic font $(\mathcal{A}, \mathcal{B}, \mathcal{C} \ldots)$ for those letters used by Viète to denote lengths of chords and line segments, variables and square-roots (see Figure 1).

2 François Viète

A significant innovation of Viète was his use of specific letters to represent variables and constants, his practice with respect to the cubic being to use the letters $\mathcal{A}$ and $\mathcal{N}$ for the unknown variable, $\mathcal{C}$ for its cube, the consonants $\mathcal{B}$ and $\mathcal{D}$ for known quantities, and occasionally $\mathcal{R}$ to denote a square-root [3, 4, 5]. Since Viète did not recognise negative roots he varies the format of his equations so they relate to the positive roots he determines. In view of this, all

[1] This minor revision of the original article corrects typographic errors, includes some of Viète's original Latin text, and incorporates some explanatory footnotes. The original published version will be available from the JSTOR archive in due course.

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the figures in the present article have been designed to illustrate the positive roots associated with Viète's cubics having the form

$$\mathcal{A}^3 - 3\mathcal{B}^2\mathcal{A} + \mathcal{B}^2\mathcal{D} = 0. \qquad (3)$$

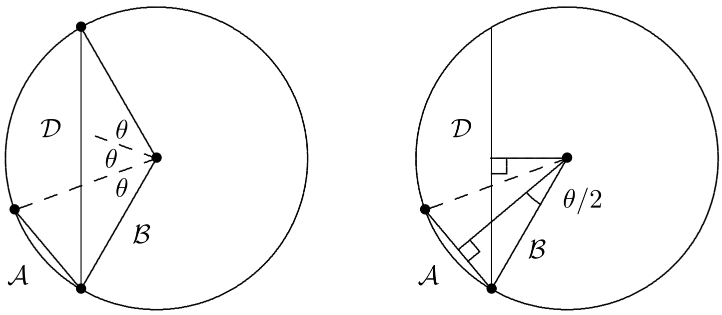

Figure 1:

For example, in chapter 6 (theorem 3) of his article entitled *On the recognition of equations* [6], published posthumously in 1615 by Alexander Anderson (1582–1620) [7], Viète describes how the chord $\mathcal{A}$ of a trisected arc associated with chord $\mathcal{D}$ of a circle of radius $\mathcal{B}$, is a solution of univariate cubics having the form $\mathcal{A}^3 - 3\mathcal{B}^2\mathcal{A} \pm \mathcal{B}^2\mathcal{D} = 0$, where $\mathcal{B} > \mathcal{D}/2$ (see Figure 1), as follows:

ALITER,[3] TERTIVM THEOREMA.

SI A cubus — B quad. 3 in A, æquetur B quad. in D, ſit autem B major D ſemiſſe: B quad. 3 in E, — E cubo æquabitur B quad. in D.

Et ſunt duo triangula rectangula æqualis B hypotenuſæ, ita ut angulus acutus ſubtenſus à perpendiculo primi, ſit triplus ad angulum acutum ſubtenſum à perpendiculo ſecundi; baſis vero dupla primi, eſt D, & ſit A dupla baſis ſecundi. E vero baſis ſimpla ſecundi,contracta,protractave longitudine ejuſ quæ poteſt quadrato triplum perpendiculi ejuſdem.

Viète (1646).[4] Ed. van Schooten, p. 91.

The English translation [5, p. 17] of the relevant parts is as follows:

If $\mathcal{A}$ cube minus $\mathcal{B}$ square thrice into $\mathcal{A}$ is equated to $\mathcal{B}$ square into $\mathcal{D}$, but $\mathcal{B}$ is greater than half $\mathcal{D}$, $\mathcal{B}$ square [thrice] into $\mathcal{E}$ minus $\mathcal{E}$ cube will be equated to $\mathcal{B}$ square into $\mathcal{D}$.
And there are two right angled triangles with equal hypotenuse $\mathcal{B}$, such that the acute angle subtended by the perpendicular of the first, is triple the acute angle subtended by the perpendicular of the second; while double the base of the first, is $\mathcal{D}$, making the double of the base of the second be $\mathcal{A}$. ...

[3] *Aliter*(L): alternative. Viète presents here what he calls an alternative version of the theorem, which he feels is somewhat clearer.

[4] From page 91 of the 1646 van Schooten edition of Viète's *Opera Mathematica* (in Latin), which is available on the web—see footnote 9 for details. I have typeset the Latin quotes here using the open-source DayRoman font http://www.ctan.org/tex-archive/fonts/DayRoman/

Viète's restriction $\mathcal{B} > \mathcal{D}/2$ is simply the condition for the cubic equation to have three *distinct* real roots. This is most easily seen by comparing coefficients between equations (1) and (3), giving $\mathcal{B}^2 = \delta^2$ and $\mathcal{D} = y_N/\delta^2$. Thus the condition $\mathcal{B} > \mathcal{D}/2$ is equivalent to $\delta > y_N/(2\delta^2)$ which, since $h = 2a\delta^3$ [2], is equivalent to $|h| > |y_N|$, namely the condition for three distinct real roots as shown in Figure 3. Geometrically of course $\mathcal{B} \geq \mathcal{D}/2$ has to apply for the chord $\mathcal{D}$ to be accommodated within a circle having radius $\mathcal{B}$, and hence for the associated cubic to be solvable trigonometrically.

Viète's approach stems from his familiarity with the then equivalent of the trigonometric triple-angle identities, since he himself had established formulae for chords of multiple arcs in terms of chords of simple arcs [8, 9], and hence he was aware that solving a cubic with three real roots was analogous to trisecting an angle. For example, if we use the identity

$$2\sin\frac{3\theta}{2} = 3\left(2\sin\frac{\theta}{2}\right) - \left(2\sin\frac{\theta}{2}\right)^3,$$

then from Figure 1 we have $2\sin\theta/2 = \mathcal{A}/\mathcal{B}$ and $2\sin 3\theta/2 = \mathcal{D}/\mathcal{B}$, and it follows that

$$\mathcal{A}^3 - 3\mathcal{B}^2\mathcal{A} + \mathcal{B}^2\mathcal{D} = 0.$$

Although Viète does not present a diagram he illustrates his approach using the equations $\mathcal{N}^3 - 300\mathcal{N} \pm 432 = 0$ [5, p. 17] (the Figure 1 equivalents have been added in square brackets). Viète continues as follows: [5]

> 1C—300 N. aquetur 432. vel etiam 300 N.—1C aquetur 432. ſunt duo triangula rectangula, quorum hypotenuſa communis eſt 10: ita ut angulus acutus primi, à perpendiculo videlicet ſubtenſus, ſit triplus ad acutum ſecundi, à ſuo queque perpendicule ſubtenſum ; baſis autem primi dupla, eſt $\frac{432}{100}$. & 1N in equalitate directe negata est baſis ſecundi: in inverſe vero negara, est baſis ſimpla ſecundi, plus minuſve ea qua poteſt quadrato triplum perpendiculum ſecundi.
>
> Conſtituta hypotenuſa communi 10. baſi ſecundi trianguli 9. ſit perpendiculum ejuſdem ſecundi $\sqrt{}$19.
>
> Primi vero hypotenuſa ſtante 10. ſit baſis $2\frac{16}{100}$. itaque quum in a hypotheſi dicetur 1 C—300 N. aquari 432. ſiet 1 N 18. vel quum dicetur 300 N.—1 C aquari 432. Fiet 1 N. 9 + $\sqrt{}$57. vel 9 —$\sqrt{}$57.
>
> Viète (1646).[6] Ed. van Schooten, p. 91.

The Schmidt translation [5, p. 17] is as follows:

> *Let* $1\mathcal{C} - 300\mathcal{N}$ *be equated to* 432*; or even let* $300\mathcal{N} - 1\mathcal{C}$ *be equated to* 432 $[300\mathcal{N} - \mathcal{N}^3 = 432]$*. There are two right triangles, of which the common hypotenuse is 10* [*radius* $\mathcal{B}$] *... Moreover double the base of the first is* $\frac{432}{100}$ [*chord* $\mathcal{D}$] *and ... the base becomes* $2\frac{16}{100}$*. ... when it is said that* $300\mathcal{N} - 1\mathcal{C}$ *is equated to* 432*,* $9 + \mathcal{R}57$ $[9 + \sqrt{57}]$ *or* $9 - \mathcal{R}57$ $[9 - \sqrt{57}]$ *will be made to be* $1\mathcal{N}$ [*chord* $\mathcal{A}$].

[5] This example was probably added by either Alexander himself (who originally published this paper of Viète's in 1615) or by van Schooten, as it appears in the van Schooten edition in italics.

[6] From page 91 of the 1646 van Schooten edition of Viète's *Opera Mathematica* (in Latin), which is available on the web—see footnote 9 for details.

In this example ($\mathcal{N}^3 - 300\mathcal{N} + 432 = 0$) the negative root is -18. Now, if one root of a reduced cubic is α, then the remaining two roots (β, γ) are [10][7]

$$\beta,\ \gamma = -\frac{\alpha}{2} \pm \frac{\sqrt{3}}{2}\sqrt{4\delta^2 - \alpha^2}$$

and since $\delta = 10$ and $\alpha = -18$ the two positive roots are

$$-\left(\frac{-18}{2}\right) \pm \frac{\sqrt{3}}{2}\sqrt{400 - 18^2} \quad = \quad 9 \pm \sqrt{57}$$

as Viète indicates.

3 René Descartes

Some years later Descartes explored Viète's approach in detail in his *La Géométrie* [11]. Indeed Descartes extended Viète's approach by showing that trisecting the arc, both internally and externally (see Figure 2) generates both of the positive roots of the reduced equation

$$z^3 - pz + q = 0. \tag{4}$$

In the English translation by Smith and Latham (12, p. 212) Descartes describes the geometric solution as follows (Descartes' original notation has been altered slightly to suit Figures 2 and 3):

> '*Finally, suppose that we have $z^3 = pz - q$. Construct the circle $OEDZ_2$ whose radius OC is equal to $\sqrt{p/3}$, and let OD, equal to $3q/p$, be inscribed in this circle; then OE, the chord of one-third the arc OED, will be the first of the required roots, and OZ_2, the chord of one-third of the other arc, will be the second.*'

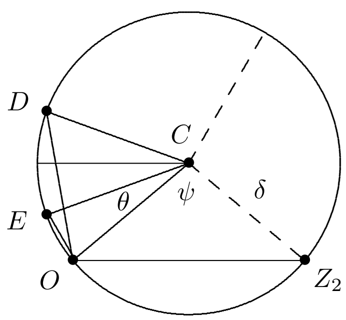

Figure 2:

[7]For derivation see: Nickalls RWD (2009). Feedback: 93.35: *The Mathematical Gazette*; 93 (March), 154–156. http://www.nickalls.org/dick/papers/maths/cubictables2009.pdf.

Comparing coefficients between equations (1) and (4) gives $p = 3\delta^2$, and so the radius of Descartes' circle ($\sqrt{p/3}$) is equivalent to δ. If θ and ψ are the internal and external trisections of the angle subtended by the chord OD then it follows that $\theta + \psi = 2\pi/3$.

It is interesting to note that Descartes is clearly just as comfortable to represent a numerical value as the length of a chord as he is to view a cube-root as the length of the side of a cube. Unfortunately for Descartes though there was no useful symbol for such a chord, as he mentions in the following passage [12, p. 216]:

> '*Indeed these terms are much less complicated than others, and they might be made even more concise by the use of some particular symbol to express such chords, just as the symbol* $\sqrt[3]{}$ *is used to represent the side of a cube.*'

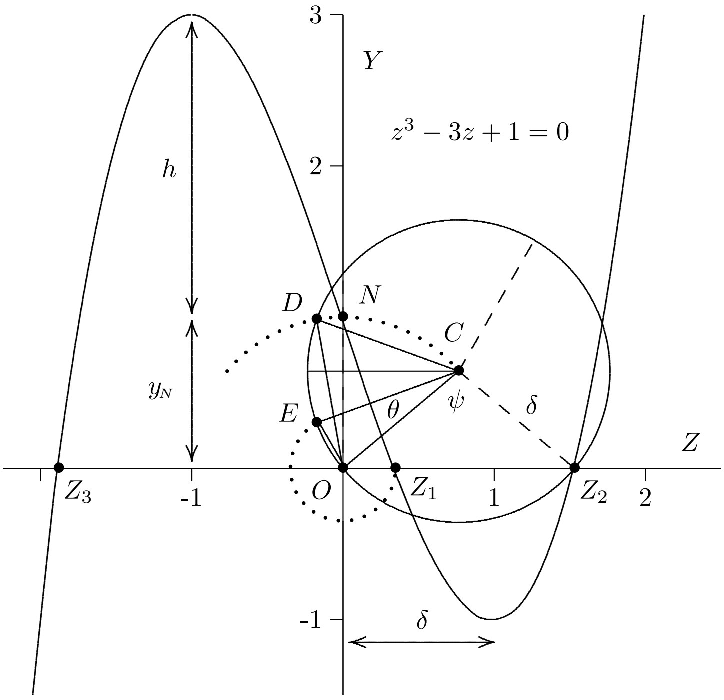

Figure 3:

4 Superposition on a cubic curve

The geometric correspondence between the chords OE and OZ_2 (Figure 2) and the positive roots can be nicely illustrated by superimposing Figure 2 on to the associated cubic curve itself. For example, in Figure 3 Descartes' figure is shown in relation to the reduced cubic equation

$$z^3 - 3z + 1 = 0 \qquad (5)$$

having roots $-1{\cdot}879$, $0{\cdot}347$, $1{\cdot}532$. Comparing equations (1) and (5) shows that $a = 1$, $\delta^2 = 1$ and $y_N = 1$, and since $h = 2a\delta^3$ it follows that in this particular case $y_N = h/2$. From equations (1) and (3) we have $\delta^2 = \mathcal{B}^2$ and $y_N = \mathcal{B}^2\mathcal{D}$, and so $\mathcal{B}^2 = 1$ and hence $y_N = \mathcal{D} \equiv OD = \delta = 1$ (see the dotted circular arc DNC). Since the chord $OD = \delta$ it follows that the triangle ODC is equilateral, and so the internal angle subtended by the chord OD is 60°, and hence θ is 20°. The chord OE is the positive root Z_1 (see the dotted circular arc between these points), which in this case is given by

$$2\delta \sin(\theta/2) = 2 \sin 10^\circ = 0{\cdot}347.$$

The external angle subtended by the chord OD is therefore 300° and so we have $\psi = 300/3 = 100^\circ$. The other positive root Z_2 is therefore given by

$$2\delta \sin(\psi/2) = 2 \sin 50^\circ = 1{\cdot}532.$$

Since we are dealing with a reduced cubic it follows that the remaining root is given by $z_3 = -(z_1 + z_2) = -1{\cdot}879$.

It is now apparent geometrically why y_N/h is such an important ratio. Since $\sin(3\theta/2) = (\mathcal{D}/2)/\mathcal{B}$ and $y_N = \mathcal{B}^2\mathcal{D}$ it follows that $\sin(3\theta/2) = y_N/(2\delta^3)$. But $h = 2a\delta^3$ [2], and since $a = 1$ it follows that $\sin(3\theta/2) = y_N/h$.

We can now compare the geometry with the standard algebraic solution (equations 2) as follows. Starting with $\sin 3\phi = y_N/h = 1/2$ we obtain $\phi = 10^\circ$ (i.e. $\phi = \theta/2$), and since $\delta = 1$ the roots are given by

$$\begin{cases} z_1 = 2\sin\phi = 0{\cdot}347, \\ z_2 = 2\sin(2\pi/3 + \phi) = 1{\cdot}532, \\ z_3 = 2\sin(4\pi/3 + \phi) = -1{\cdot}879. \end{cases}$$

A significant advantage of using equations with geometric coefficients is that it is then easy to visualise the curve associated with a given equation, and conversely. For example, we have needed to work throughout with reduced cubic equations having two positive real roots in order to illustrate the methods of Viète and Descartes. Assuming that the leading coefficient is positive, then we can easily 'see' from equation (1) (and Figure 3) that such equations must have a negative z coefficient (forces $\delta^2 > 0$ and so δ is real), a positive constant term ($y_N \geq 0$ to force two positive roots), and for $0 \leq y_N/h \leq 1$ (to force two positive and real roots).

In conclusion, I would like to suggest that a simple approach for merging the algebra and geometry of polynomials in an interesting way, which greatly enhances visualisation, is to use equations with coefficients consisting of geometric objects, as illustrated by equation (1) [2].

5 References

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[8] see [6].

[9] This is included in the 1646 edition of Viète's *Opera mathematica* (collected works). Edited by Frans van Schooten (Leiden 1646). [facsimile available at: http://visualiseur.bnf.fr/ark:/12148/bpt6k107597d]; (Georg Olms Verlag, Hildesheim (reprint) 1970). A copy is held in British Library, London. The Latin extracts in this paper are from page 91.

[10] see English translation by Ian Bruce available from http://www.17centurymaths.com/contents/Angular%20Sections.pdf.

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