

Feedback—On Note 92.35

Real roots of cubics: explicit formula for quasi-solutions

RWD Nickalls¹

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<http://www.nickalls.org/dick/papers/math/cubictables2009.pdf>

RWD Nickalls writes: I was very interested in the way Edgar Rechtschaffen² tabulated the roots of the cubic $w^3 - 3w + 2\alpha = 0$ in terms of the parameter α , since until the advent of personal computers, cubics were commonly solved using one of the many available tables [1] developed for solving equations numerically.

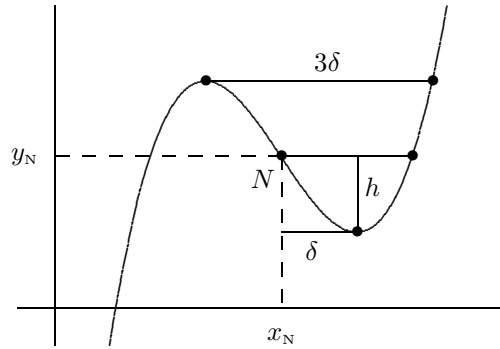


Figure 1:

In fact the reason such a tabulation can be based on Rechtschaffen's parameter α lies in a generally overlooked geometric feature of the cubic $f(x) \equiv ax^3 + bx^2 + cx + d$, namely that $h = 2a\delta^3$ (see figure 1 [3]). This relationship immediately suggests an elegant solution [2] to the (now historical) problem of tabulating the roots of the reduced cubic in a simple generic form (see Table 1), since dividing the cubic's reduced geometric form [3] $az^3 - 3a\delta^2z + y_N = 0$ throughout by h (excluding the trivial $h = \delta = 0$ case), yields the useful normalised form

$$\left(\frac{z}{\delta}\right)^3 - 3\left(\frac{z}{\delta}\right) + 2\frac{y_N}{h} = 0. \quad (1)$$

Thus α is revealed to be the ratio y_N/h , and we can begin to see why this method works.

¹Department of Anaesthesia, Nottingham University Hospitals, City Hospital Campus, Nottingham, UK. email: dick@nickalls.org

² E. Rechtschaffen (2008). NOTE 92.35 Real roots of cubics: explicit formula for quasi-solutions. *The Mathematical Gazette*; **92** (July 2008), pp. 268–276 (JSTOR).

If $f(x) = 0$ is the cubic to be solved then the roots z_i ($i = 1, 2, 3$) of its reduced form can be expressed as multiples of δ (equivalent to what Rechtschaffen terms x_m), i.e., $x_i = x_N + z_i = x_N + \delta(z_i/\delta)$ where [3] $x_N = -b/(3a)$ and $\delta^2 = (b^2 - 3ac)/(9a^2)$. Thus Rechtschaffen's tabulated values of $r_i(\alpha)$ for values of α are exactly equivalent to the more intuitive z_i/δ associated with y_N/h , as shown in Table 1 below.

y_N/h	z_1/δ	z_2/δ	z_3/δ
± 2.00	∓ 2.195823345446	$\pm 1.097911672723 \pm j 0.785003263244$	
± 1.75	∓ 2.151069024856	$\pm 1.075534512428 \pm j 0.685801328573$	
± 1.50	∓ 2.103803402736	$\pm 1.051901701368 \pm j 0.565235851677$	
± 1.25	∓ 2.053621575879	$\pm 1.026810779106 \pm j 0.403758818630$	
± 1.00	∓ 2	± 1	± 1
± 0.75	∓ 1.942241850970	± 0.557874698332	± 1.384367152638
± 0.50	∓ 1.879385241572	± 0.347296355334	± 1.532088886238
± 0.25	∓ 1.810037929234	± 0.168254401781	± 1.641783527453
0	$-\sqrt{3}$	0	$\sqrt{3}$

Table 1: (modified from [2])

Finally, Rechtschaffen's lemma 1 stems from the fact that any one of the roots z_i , say z_1 , can be represented by $z_1 = p + p'$ where $pp' = \delta^2$, and hence the remaining two roots are $p\omega + p'\omega^2$, $p\omega^2 + p'\omega$, where ω is a complex cube root of unity, as follows.

$$\begin{aligned}
 z_2, z_3 &= p \left(-\frac{1}{2} \pm j \frac{\sqrt{3}}{2} \right) + p' \left(-\frac{1}{2} \mp j \frac{\sqrt{3}}{2} \right), \\
 &= -\frac{p+p'}{2} \pm j \frac{\sqrt{3}}{2} \sqrt{(p+p')^2 - 4pp'}, \\
 &= -\frac{z_1}{2} \pm \frac{\sqrt{3}}{2} \sqrt{4\delta^2 - z_1^2}.
 \end{aligned}$$

Thus given *any* root of (1), say z_1/δ , then the remaining two roots are given by

$$\frac{z_2}{\delta}, \frac{z_3}{\delta} = -\frac{z_1}{2\delta} \pm \frac{1}{2} \sqrt{12 - 3 \left(\frac{z_1}{\delta} \right)^2}.$$

Example

Solve $f(x) \equiv x^3 - 9x^2 + 15x + 17 = 0$

The key parameters are: $a = 1$, $x_N = 3$, $y_N = f(x_N) = 8$, $\delta = 2$, $h = 2a\delta^3 = 16$. Thus $y_N/h = 0.5$ and so using Table 1 the roots $x_i = x_N + \delta(z_i/\delta)$ are given by

$$x_1 = 3 + 2(-1.879385241572) = -0.7587,$$

$$x_2 = 3 + 2(0.347296355334) = 3.6945,$$

$$x_3 = 3 + 2(1.532088886238) = 6.0641.$$

References

1. Salzer H.E., Richards C.H. and Arsham I. *Table for the solution of cubic equations*. pp. 161. (McGraw-Hill Book Company Inc., New York, London, 1958)
2. R. W. D. Nickalls. Solving the cubic using tables. *Theta*; **10** (No. 2, Autumn, 1996) pp. 21–24. (Pub: Mathematics Department, Manchester Metropolitan University, UK) [ISSN: 0953–0738]
<http://www.nickalls.org/dick/papers/math/cubictables1996.pdf>
3. R. W. D. Nickalls. A new approach to solving the cubic: Cardan's solution revealed. *Math. Gaz.*, **77** (November 1993) pp. 354–359 (JSTOR).
<http://www.nickalls.org/dick/papers/math/cubic1993.pdf>
<http://www.jstor.org/stable/3619777>

