Solving the cubic using tables [1]

RWD Nickalls [2]

Theta [3] (1996); vol. 10 (No. 2, Autumn), p. 21–24.
http://www.nickalls.org/dick/papers/maths/cubictables1996.pdf

1 Introduction

Recent articles have shown that using certain new parameters related to the geometry of the cubic (x_N, y_N, h, δ—see Table 1) not only reveals how the solution is related to the geometry, but also greatly facilitates the solution itself (Nickalls 1993, 1996).

This article shows how these new parameters also enable the solution to the cubic to be tabulated. This method of using parameters related to the geometry may have an application to other polynomials.

For example, the general cubic $f(x) = ax^3 + bx^2 + cx + d = 0$ having roots x_1, x_2, x_3, can be regarded as having a reduced form

$$az^3 - 3a\delta^2 z + y_N = 0, \tag{1}$$

with roots (z_1, z_2, z_3) given by

$$\begin{cases} z_1 = x_1 - x_N, \\ z_2 = x_2 - x_N, \\ z_2 = x_3 - x_N. \end{cases}$$

where

$$\begin{aligned} \delta^2 &= (b^2 - 3ac)/9a^2, \\ h &= 2a\delta^3, \\ x_N &= -b/3a, \\ y_N &= f(x_N). \end{aligned}$$

[1] This minor revision of the original article corrects some typographic errors. Some aspects of this paper appeared in Nickalls RWD (2009). Feedback: 93.35: *The Mathematical Gazette*; 93 (March), 154–156. http://www.nickalls.org/dick/papers/maths/cubictables2009.pdf).

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[3] The journal *Theta* (ISSN 0953-0738) ceased publication in 1996. It was published twice a year (in Spring and Autumn) by the Department of Mathematics, Crewe and Alsager Faculty, Manchester Metropolitan University, Crewe, Cheshire, CW1-5DU, UK.

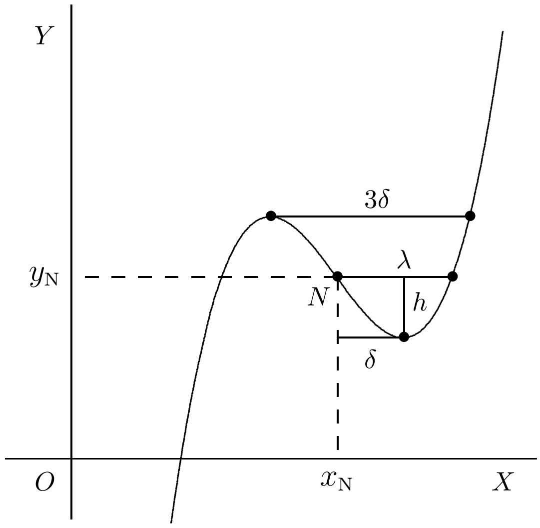

Figure 1:
Cubic showing the new parameters x_N, y_N, h, δ. N is the point of symmetry.

2 A general solution in Table form

These new parameters also allow the roots of the reduced cubic to be tabulated in terms of the parameter δ as y_N/h is a function of z/δ, since re-arranging Equation 1 gives

$$y_N = 3a\delta^2 z - az^3$$

and since $h = 2a\delta^3$ then

$$\left(\frac{y_N}{h}\right) = \frac{1}{2}\left\{3\left(\frac{z}{\delta}\right) - \left(\frac{z}{\delta}\right)^3\right\}$$

Since the discriminant of the cubic is $y_N^2 - h^2$ (see Nickalls, 1993, 1996b), it follows that y_N/h also discriminates between the cases of one real root ($|y_N| > |h|$) and three real roots ($|y_N| \leq |h|$)—see Figure 1. The case of three equal real roots ($h = y_N = 0$) is trivial as then $\delta = 0$.

It follows therefore that a table of general solutions can be constructed for a range of values of y_N/h, so that by entering the value of y_N/h for a given cubic in the left-hand column, the values of z_1/δ, z_2/δ, z_3/δ, can be extracted directly. An extract from such a table is shown in Table 1.

3 Constructing the table

In order to facilitate the construction of the Table, it is necessary to express z/δ in terms of y_N/h for both the single-real-root case and the three-real-root case, as follows. (see Nickalls (1993) for details of the following equations).

y_N/h	z_1/δ	z_2/δ	z_3/δ
···	....	....	
±1·05	∓2·01102987	$\pm(1{\cdot}00551493 \pm j0{\cdot}18215614)$	
±1·04	∓2·00883677	$\pm(1{\cdot}00441838 \pm j0{\cdot}16299963)$	
±1·03	∓2·00663728	$\pm(1{\cdot}00331864 \pm j0{\cdot}14122639)$	
±1·02	∓2·00443136	$\pm(1{\cdot}00221568 \pm j0{\cdot}11536388)$	
±1·01	∓2·00221896	$\pm(1{\cdot}00110948 \pm j0{\cdot}08161238)$	
±1	∓2	±1	±1
±0·99	∓1·99777447	±0·91719968	±1·08057478
±0·98	∓1·99554232	±0·88219381	±1·11334851
±0·97	∓1·99330348	±0·85503280	±1·13827067
±0·96	∓1·99105788	±0·83192461	±1·15913326
±0·95	∓1·98880550	±0·81140139	±1·17740410
···	....	....	....
±0·55	∓1·89242324	±0·38580936	±1·50661388
±0·54	∓1·88983562	±0·37800426	±1·51183136
±0·53	∓1·88723813	±0·37025254	±1·51698558
±0·52	∓1·88463068	±0·36255211	±1·52207856
±0·51	∓1·88201309	±0·35490071	±1·52711237
±0·50	∓1·87938536	±0·34729670	±1·53208865
···	....	....	....
±0·05	∓1·74848312	±0·03334616	±1·71513695
±0·04	∓1·74523347	±0·02667342	±1·71856004
±0·03	∓1·74196572	±0·02000309	±1·72196262
±0·02	∓1·73867961	±0·01333459	±1·72534500
±0·01	∓1·73537480	±0·00666723	±1·72870756
±0	$-\sqrt{3}$	0	$\sqrt{3}$

Table 1. This table gives the roots z/δ for various values of y_N/h. When $0 \le |y_N/h| \le 1$ there are three real roots (two equal when $|y_N/h| = 1$). When $|y_N/h| > 1$ there is one real root and two complex roots. Care is needed to select the appropriate sign for the roots, namely that corresponding to the sign of y_N/h.

3.1 Three real roots

This category relates to the following range

$$-1 \leq \left(\frac{y_N}{h}\right) \leq +1.$$

In this case the trigonometric solution is $z = 2\delta cos\theta$ where $cos3\theta = -y_N/h$, in which case $z/\delta = 2cos\theta$. One can therefore determine the three values of (z/δ) associated with any particular value of y_N/h.

3.2 One real root

This category relates to the following range

$$\left|\frac{y_N}{h}\right| > 1$$

The real root z is therefore given by

$$z = \sqrt[3]{\frac{1}{2a}\left(-y_N + \sqrt{y_N^2 - h^2}\right)} + \sqrt[3]{\frac{1}{2a}\left(-y_N - \sqrt{y_N^2 - h^2}\right)}$$

where $y^2 - h^2$ is the discriminant of the cubic. Since $2a\delta^3 = h$, it follows that z/δ is given by

$$\frac{z}{\delta} = \sqrt[3]{\frac{-y_N}{h} + \sqrt{\left(\frac{y_N}{h}\right)^2 - 1}} + \sqrt[3]{\frac{-y_N}{h} - \sqrt{\left(\frac{y_N}{h}\right)^2 - 1}}\,.$$

Furthermore, if *any* one of the roots of a reduced cubic (Equation 1) is z, then it can be shown[4] that the other two roots are given by

$$-\frac{z}{2} \pm \sqrt{3\delta^2 - 3\left(\frac{z}{2}\right)^2}.$$

Thus if z/δ is the only real root of a reduced cubic, then the two associated complex roots are given by

$$-\frac{z}{2\delta} \pm j\sqrt{3\left(\frac{z}{2\delta}\right)^2 - 3}$$

where $j^2 = -1$. Thus for the case when there is only one real root the table can be easily extended to include the complex roots as well.

Note that since the coefficient of the z^3 term in Equation 1 is incorporated in the variable h, it is only necessary to consider the sign of y_N/h when selecting the appropriate signs of the roots from Table 1.

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[4] Nickalls (2009), Feedback: 93.35, *The Mathematical Gazette*; 93 (March), 154–156 (http://www.nickalls.org/dick/papers/maths/cubictables2009.pdf), where we express it in the more understandable form $-\frac{z}{2} \pm \frac{\sqrt{3}}{2}\sqrt{4\delta^2 - z^2}$

4 Example

Consider the cubic $x^3 - 7x^2 + 14x - 8 = 0$ (roots 1, 2, 4) the derived parameters are [5]

$$\begin{aligned} x_N &= 7/3 \\ \delta^2 &= 7/9 \\ \frac{y_N}{h} &= -0{\cdot}5399 \end{aligned}$$

If the roots of the reduced form are z_1, z_2, z_3, then entering $-0{\cdot}54$ in the left-hand column (Table 1) as the nearest value to the given y_N/h and taking care to select the appropriate signs, gives the following three values

$$\begin{aligned} z_1/\delta &= +1{\cdot}88983562 \\ z_2/\delta &= -0{\cdot}37800426 \\ z_3/\delta &= -1{\cdot}51183136 \end{aligned}$$

Since $x = x_N + z$, then the roots of the original equation are given by

$$\begin{aligned} x_1 &= x_N + \delta\,(+1{\cdot}88983552) &= 4{\cdot}0000116 \\ x_2 &= x_N + \delta\,(-0{\cdot}37800394) &= 1{\cdot}9999649 \\ x_3 &= x_N + \delta\,(-1{\cdot}51183157) &= 1{\cdot}0000233 \end{aligned}$$

In practice, a table listing y_N/h to only two decimal places is quite adequate, since this still gives a very good approximation ($\pm 0{\cdot}005\delta$) when the given y_N/h falls between tabulated values.

5 References

1. Nickalls, RWD (1993). A new approach to solving the cubic: Cardan's solution revealed. *Mathematical Gazette*, **77** (November), 354–359 (JSTOR). http://www.nickalls.org/dick/papers/maths/cubic1993.pdf

2. Nickalls, RWD (1996). A note on solving cubics. *Mathematical Gazette*, **80** (November), 576–577 (JSTOR). http://www.nickalls.org/dick/papers/maths/cubefink.pdf

3. Nickalls, RWD and Dye RH (1996). The geometry of the discriminant of a polynomial. *Mathematical Gazette*, **80** (July), 297–285 (JSTOR). http://www.nickalls.org/dick/papers/maths/discriminant1996.pdf

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[5] $y_N = F(x_N) = -20/27$, and $h = 2a\delta^3 = 14\sqrt{7}/27$