

# A note on solving cubics <sup>1</sup>

RWD Nickalls <sup>2</sup>

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<http://www.nickalls.org/dick/papers/maths/cubefink.pdf>

In a recent note in the March 1995 issue of the *Gazette Fink* [1] discussed an interesting problem involving a dipstick for a hemispherical tank, which involved inverting a cubic.

An alternative and instructive avenue of approach for solving this cubic is as follows, which illustrates the method described in my article [2]. This is a very simple and intuitive method which uses parameters linked to the geometry of the cubic ( $x_N$ ,  $y_N$ ,  $\delta$ ,  $h$  etc.), and generally allows the roots to be written down almost immediately with a minimum of working. The trick is to visualise the geometry while noticing the relative magnitude of  $|y_N|$  and  $|h|$ , and then everything becomes clear.

For example, this particular problem involved the following ‘reduced’ cubic  $f(1-b) = 0$ .

$$(1-b)^3 - 3(1-b) - 2(a-1) = 0 \quad \begin{cases} 0 \leq a \leq +1 \\ 0 \leq b \leq +1 \end{cases} \quad (1)$$

Using the notation described in [2] and comparing Equation 1 with my standard form of the reduced cubic  $az^3 - 3a\delta^2z + y_N = 0$  yields  $y_N = -2(a-1)$  and  $\delta^2 = 1$ , and so  $h = -2$  (since  $h = -2a\delta^3$ ). Since in this case  $y_N^2 < h^2$  (because  $0 \leq a \leq +1$ ) it follows that Equation 1 must have three real roots (see Figure 1) which can be immediately written down using the formula for the case of three real roots [2] as follows.

$$(1-b) = x_N + 2\delta \cos \theta \quad \text{where} \quad \cos 3\theta = \frac{y_N}{h} = (a-1)$$

Since  $x_N = 0$  (because Equation 1 is already in the reduced form) and  $|\delta| = 1$ , the three roots  $b_1, b_2, b_3$  are therefore given by

$$\begin{aligned} 1-b_1 &= 2 \cos \theta \\ 1-b_2 &= 2 \cos \left( \frac{2\pi}{3} - \theta \right) \\ 1-b_3 &= 2 \cos \left( \frac{2\pi}{3} + \theta \right) \end{aligned}$$

In practice only one of the roots is relevant in this case, and again the geometry makes it clear which root is required, as follows. Since  $0 \leq (1-b) \leq +1$  (because  $0 \leq b \leq +1$ ) and  $|\delta| = 1$ , it follows immediately that the relevant solution is  $b_2$  since  $-\delta \leq (1-b_2) \leq +\delta$ , as this range includes the critical range 0 to +1.

<sup>1</sup>This minor revision corrects some typographical errors. The original published version is available at <http://www.jstor.org/stable/3618532>

<sup>2</sup>Department of Anaesthesia, Nottingham University Hospitals, City Hospital Campus, Nottingham, UK.  
email: [dick@nickalls.org](mailto:dick@nickalls.org)

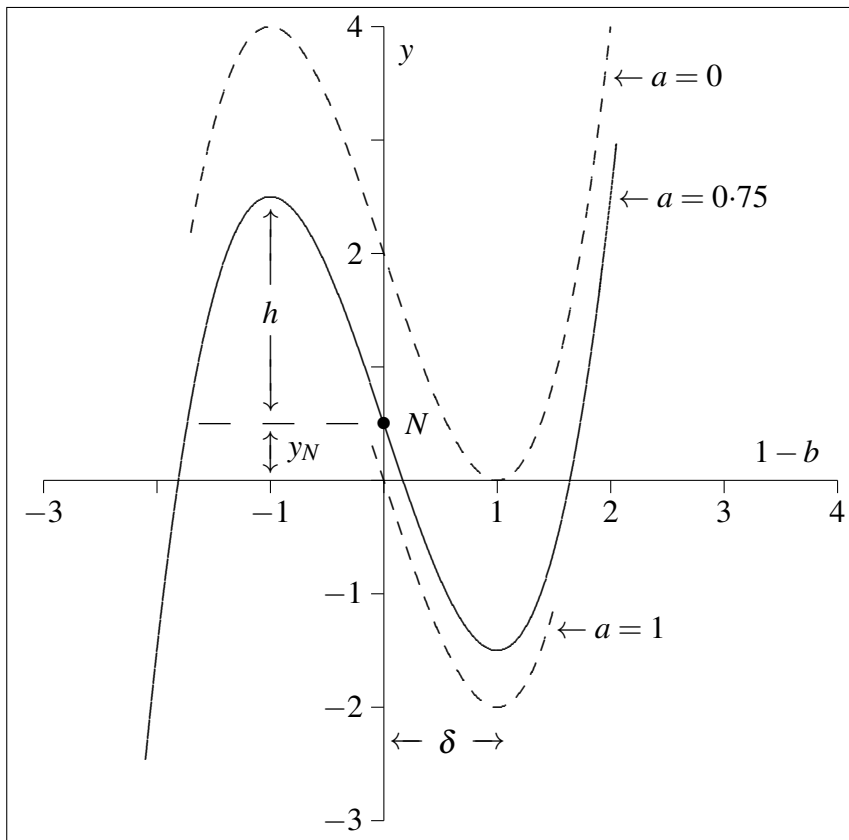


Figure 1:

It turns out that the graph of  $b$  against  $a$  becomes more linear as  $a$  increases. That this is so can be seen by looking at the original cubic (Figure 1), and noticing that the root  $1 - b_2$  lies on the relatively linear central portion of the cubic for much of the range of  $y_N$  ( $0 \leq y_N \leq +2$ ).

If  $a = 0$  (i.e.  $y_N = +2$ ) then  $y_N^2 = h^2$  and therefore there is a double real root at  $+\delta$  ( $= +1$ ). As  $a$  increases ( $y_N$  decreases) then the root  $1 - b_2$  lies on a progressively more linear part of the cubic, and so the relationship between  $y_N$  and  $1 - b_2$  (and hence between  $a$  and  $b$ ) also becomes progressively more linear, until eventually the cubic's point of symmetry ( $N$ ) lies on the  $1 - b$  axis (i.e. when  $a = 1, b = 1$ ).

I would like to suggest therefore, that students are encouraged to see a cubic in terms of its anatomical parts ( $x_N, y_N, \delta, h$ ), since this then greatly facilitates the solution.

## References

1. Fink, A. M. (1995). A dipstick for a hemispherical tank. *Mathematical Gazette*. 79 (March), pp. 115–117 (JSTOR).
2. Nickalls, R. W. D. (1993). A new approach to solving the cubic: Cardan's solution revealed. *Mathematical Gazette*. 77 (November), pp. 354–359 (JSTOR).  
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