

A note on solving cubics

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<http://www.nickalls.org/dick/papers/maths/cubefink.pdf>

In a recent note in the March 1995 issue of the *Gazette Fink* [1] discussed an interesting problem involving a dipstick for a hemispherical tank, which involved inverting a cubic.

An alternative and instructive avenue of approach for solving this cubic is as follows, which illustrates the method described in my article [2]. This is a very simple and intuitive method which uses parameters linked to the geometry of the cubic (x_N , y_N , δ , h etc.), and generally allows the roots to be written down almost immediately with a minimum of working. The trick is to visualise the geometry while noticing the relative magnitude of $|y_N|$ and $|h|$, and then everything becomes clear.

For example, this particular problem involved the following ‘reduced’ cubic $f(1-b) = 0$.

$$(1-b)^3 - 3(1-b) - 2(a-1) = 0 \quad \begin{cases} 0 \leq a \leq +1 \\ 0 \leq b \leq +1 \end{cases} \quad (1)$$

Using the notation described in [2] and comparing Equation 1 with my standard form of the reduced cubic $az^3 - 3a\delta^2z + y_N = 0$ yields $y_N = -2(a-1)$ and $\delta^2 = 1$, and so $|h| = 2$ (since $h = 2a\delta^3$). Since in this case $|y_N| < |h|$ (because $0 \leq a \leq +1$) it follows that Equation 1 must have three real roots (see Figure 1) which can be immediately written down using the formula for the case of three real roots [2] as follows.

$$(1-b) = x_N + 2\delta \cos \theta \quad \text{where} \quad \cos 3\theta = \frac{-y_N}{|h|} = (a-1)$$

Since $x_N = 0$ (because Equation 1 is already in the reduced form) and $|\delta| = 1$, the three roots b_1, b_2, b_3 are therefore given by

$$\begin{aligned} 1 - b_1 &= 2 \cos \theta \\ 1 - b_2 &= 2 \cos \left(\frac{2\pi}{3} - \theta \right) \\ 1 - b_3 &= 2 \cos \left(\frac{2\pi}{3} + \theta \right) \end{aligned}$$

In practice only one of the roots is relevant in this case, and again the geometry makes it clear which root is required, as follows. Since $0 \leq (1-b) \leq +1$ (because $0 \leq b \leq +1$) and $|\delta| = 1$, it follows immediately that the relevant solution is b_2 since $-\delta \leq (1-b_2) \leq +\delta$, as this range includes the critical range 0 to +1.

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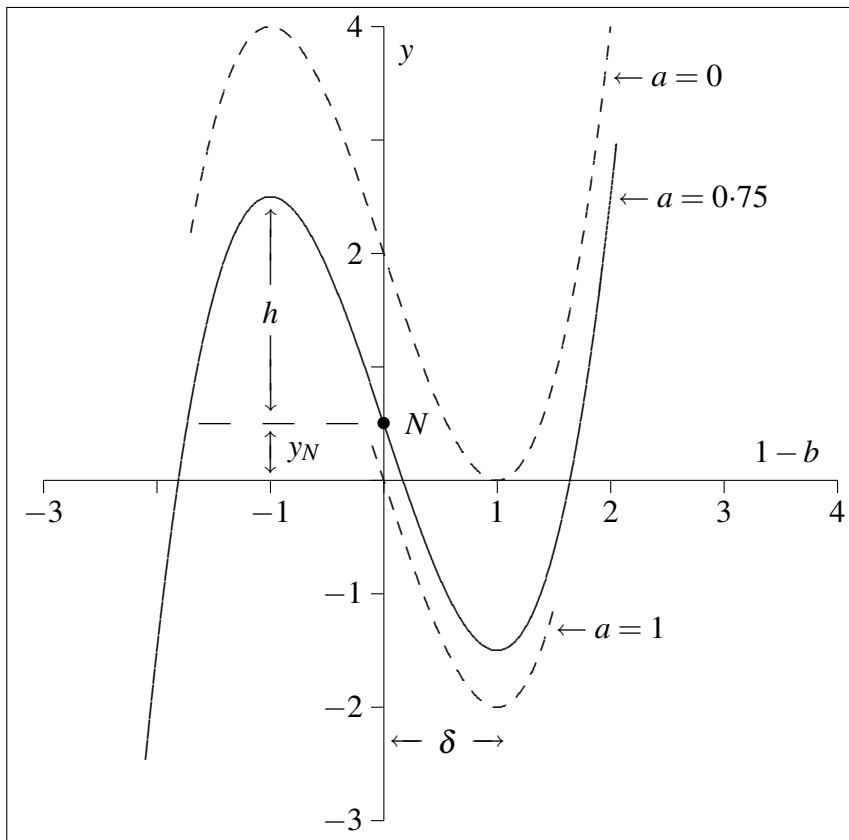


Figure 1:

It turns out that the graph of b against a becomes more linear as a increases. That this is so can be seen by looking at the original cubic (Figure 1) and noticing that the root $1 - b_2$ lies on the relatively linear central portion of the cubic for much of the range of y_N ($0 \leq y_N \leq +2$).

If $a = 0$ (i.e. $y_N = +2$) then $|y_N| = |h|$ and therefore there is a double real root at $+\delta$ ($= +1$). As a increases (y_N decreases) then the root $1 - b_2$ lies on a progressively more linear part of the cubic, and so the relationship between y_N and $1 - b_2$ (and hence between a and b) also becomes progressively more linear, until eventually the cubic's point of symmetry (N) lies on the $1 - b$ axis (i.e. when $a = 1, b = 1$).

I would like to suggest therefore, that students are encouraged to see a cubic in terms of its anatomical parts (x_N, y_N, δ, h), since this then greatly facilitates the solution.

References

1. Fink, A. M. (1995). A dipstick for a hemispherical tank. *Mathematical Gazette*. **79** (March), pp. 115–117 (JSTOR).
2. Nickalls, R. W. D. (1993). A new approach to solving the cubic: Cardan's solution revealed. *Mathematical Gazette*. **77** (November), pp. 354–359 (JSTOR).
<http://www.nickalls.org/dick/papers/math/cubic1993.pdf>